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HAUSDORFF DIMENSION OF SATURATED SETS FOR DIFFEOMORPHISMS WITH DOMINATED SPLITTING

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ABSTRACT. Let f be a diffeomorphism of a manifold having a compact invariant set with dominated splitting. Some lower and upper bounds on the Hausdorff dimension of saturated sets are given in terms of the Lyapunov exponents and the entropy.

Let M be a compact metric space and $f : M \to M$ be a continuous map of M. Given a continuous function $\varphi : M \to \mathbb{R}$, we consider the set

$$K_\alpha = \left\{ x \in M \ : \ \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) = \alpha \right\}$$

for $\alpha \in \mathbb{R}$, which is called the level set of the Birkhoff averages of φ. The Birkhoff ergodic theorem tells us that when μ is an ergodic f-invariant probability measure, the μ-measure of K_α is equal to 1 for $\alpha = \int \varphi d\mu$, and 0 for any other $\alpha \in \mathbb{R}$. However, the multifractal analysis assures that for several important dynamical systems f and generic φ, there exist uncountably many values of α such that the 'sizes' of K_α are not small in terms of the dimension and of the entropy (see, e.g., [4], [8], [12], [13]). For example, if M is a repeller of an expanding, $C^{1+\delta}$-conformal mixing map f, then the following equation holds:

$$\dim_H(K_\alpha) = \max\left\{ \frac{h_\mu(f)}{\int \log \|D_x f\| d\mu} \ : \ \int \varphi d\mu = \alpha \right\} \tag{0.1}$$

where $h_\mu(f)$ is the measure theoretical entropy of μ, $\|D_x f\|$ is the operator norm of the differential $D_x f$, and $\dim_H$ is the Hausdorff dimension ([8]).

In this paper we deal with the Hausdorff dimension of some saturated sets for diffeomorphisms having a compact invariant set with dominated splitting. Our purpose here is twofold: Firstly, we consider saturated sets instead of the level sets of the Birkhoff averages. Secondly, we get rid of the assumptions of the uniform hyperbolicity (or expansion) and the conformality of f. From now on we consider a C^2 diffeomorphism $f : M \to M$ of a compact smooth Riemannian manifold M.

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To investigate time averages along orbits we introduce the *empirical measure* of order n of $x \in M$, which is defined by

$$\delta_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)},$$

where δ_y is the Dirac measure at $y \in M$. And we denote as $V(x)$ the limit-point set of the sequence $\{\delta_n(x)\}_{n\in\mathbb{N}}$ in the collection $\mathcal{M}(M)$ of all probability measures on M. A subset $D \subset M$ is said to be *saturated* if $x \in M$ satisfies $V(x) = V(y)$ for some $y \in D$, then $x \in D$. We remark that the level set K_α is saturated. This is a simple consequence of the fact that $x \in K_\alpha$ if and only if every $\mu \in V(x)$ satisfies $\int \varphi d\mu = \alpha$. In this paper we shall consider more general saturated sets defined by

$$G(K) = \{x \in M : V(x) \subset K\}$$

for some closed set K in the collection $\mathcal{M}_f(M)$ of all f-invariant probability measures on M. With this notation we can write

$$K_\alpha = G\left(\left\{\mu \in \mathcal{M}_f(M) : \int \varphi d\mu = \alpha\right\}\right).$$

In the case when $K = \{\mu\}$, we write simply G_μ.

A compact f-invariant set Λ is said to be an *isolating set* if there is an open neighborhood $U \supset \Lambda$ (called an *isolating block*) such that $\Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$. We say that an isolating set Λ admits a *dominated splitting* if there exist a continuous Df-invariant splitting $E^{cs} \oplus E^{cu}$ of the tangent bundle of M over Λ and a constant $0 < \lambda < 1$ satisfying

$$\|Df|E^{cs}(x)\| \cdot \|(Df|E^{cu}(x))^{-1}\| \le \lambda,$$

for all $x \in \Lambda$ and $n \ge 1$. Moreover, to avoid complication we impose additional assumptions as follows:

(1) if $x \in U$ and $f(x) \notin U$ then $f^n(x) \notin U$ for every $n \in \mathbb{N}$, and
(2) the dimension of $E^{cs}(x)$ does not depend on $x \in \Lambda$.

Hereafter, the dimension of E^{cs} will be denoted by d^s. The domination condition of the splitting is a weaker form of the uniform hyperbolicity, and its statistical properties were intensively studied in several papers (cf. [1], [2], [5], [7], [14]). On the other hand, to the best of our knowledge, the multifractal analysis has not been studied under the domination condition.

In the present paper we consider a new class of invariant measures instead of hyperbolic measures. Let $\mathcal{M}_f(\Lambda)$ be the set of all f-invariant probability measures on an isolating set Λ with dominated splitting. For $\mu \in \mathcal{M}_f(\Lambda)$, by the Kingman sub-additive ergodic theorem [10], the following limits exist for μ-almost every $x \in M$:

$$\chi_1(x) = \lim_{n\to\infty}\frac{1}{n}\log\|D_x f^n|E^{cu}\|,$$

$$\chi_c(x) = \lim_{n\to\infty}\frac{1}{n}\log\|D_{f^n x} f^{-n}|E^{cu}\|^{-1},$$

$$\chi_s(x) = \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n | E^{cs}\|.$$

Using these characteristics we define a subset $\mathcal{H}_f(\Lambda)$ of $\mathcal{M}_f(\Lambda)$ as follows:

$$\mathcal{H}_f(\Lambda) = \{\mu \in \mathcal{M}_f(\Lambda) \ : \ \chi_c(\mu) > 0 > \chi_s(\mu)\}.$$

Here we set

$$\chi_\sigma(\mu) = \int_M \chi_\sigma(x) d\mu(x) \ (\sigma = 1, c, s).$$

Then we have the following:

Theorem 0.1. *$\mathcal{H}_f(\Lambda)$ is open in $\mathcal{M}_f(\Lambda)$.*

To state our main theorem, we use the notion of topological entropy of non-compact sets which was defined by Bowen ([6]). Recently Pfister and Sullivan [12] showed that

$$\sup\{h_{\text{top}}(f, G_\mu) : \mu \in K\} \le h_{\text{top}}(f, G(K)) \le \sup\{h_\mu(f) : \mu \in K\},$$

where $h_{\text{top}}(f, Z)$ is the topological entropy of $Z \subset M$. Our main theorem gives lower and upper bounds on the Hausdorff dimension of $G(K)$ as follows:

Theorem 0.2. *Let $f \colon M \to M$ be a C^2 diffeomorphism exhibiting an isolating set Λ with a dominated splitting which satisfies the conditions* (1) *and* (2). *If K is a closed subset contained in $\mathcal{H}_f(\Lambda)$ and satisfies that $G_\mu \ne \emptyset$ for some $\mu \in K$, then we have*

$$d^s + \sup_{\mu\in K} \left\{ \frac{h_{\text{top}}(f, G_\mu)}{\chi_1(\mu)} \right\} \le \dim_H G(K) \le d^s + \sup_{\mu \in K} \left\{ \frac{h_\mu(f)}{\chi_c(\mu)} \right\}.$$

In the case when $K = \{\mu\}$, we can obtain an upper bound by using the topological entropy of G_μ.

Theorem 0.3. *Let f and Λ be as in Theorem 0.2. For $\mu \in \mathcal{H}_f(\Lambda)$ with $G_\mu \ne \emptyset$, we have*

$$d^s + \frac{h_{\text{top}}(f, G_\mu)}{\chi_1(\mu)} \le \dim_H G_\mu \le d^s + \frac{h_{\text{top}}(f, G_\mu)}{\chi_c(\mu)}.$$

By the result of [9] we can give a sufficient condition for the equality to hold.

Theorem 0.4. *Let f and Λ be as in Theorem 0.2. If K is a closed subset of $\mathcal{H}_f(\Lambda)$ such that for every $\mu \in K$*

(a) $\chi_1(\mu) = \chi_c(\mu)$ *and*

(b) *μ is hyperbolic and satisfies the almost transversality condition,*

then we have

$$\dim_H G(K) = d^s + \sup_{\mu\in K} \left\{ \frac{h_\mu(f)}{\chi_1(\mu)} \right\}.$$

Here the hyperbolicity and the almost transversality of invariant measures are defined as follows: A point $x \in \Lambda$ is said to be *Lyapunov regular* if there exist real numbers $\chi_1(x) > \chi_2(x) > \cdots > \chi_{r(x)}(x)$ and a $D_x f$-invariant decomposition $T_x M = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for each $i = 1, 2, \ldots, r(x)$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \chi_i(x) \qquad (v \in E_i(x) \setminus \{0\})$$

exists, and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det(D_x f^n)| = \sum_{i=1}^{r(x)} \chi_i(x) \dim E_i(x).$$

By the multiplicative ergodic theorem ([11]) Γ has full μ-measure. The numbers $\chi_i(x)$ are called the *Lyapunov exponents* of f at the point x. We call the measure μ *hyperbolic* if none of the Lyapunov exponents for μ vanish and there exist Lyapunov exponents with different signs for μ-almost everywhere.

For $x \in \Gamma$, we define the *unstable* and *stable manifolds* at x as

$$\mathcal{W}^u(x) = \left\{ y \in M \colon \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0 \right\},$$

$$\mathcal{W}^s(x) = \left\{ y \in M \colon \limsup_{n\to\infty} \frac{1}{n} \log d(f^{n}(x), f^{n}(y)) < 0 \right\},$$

where d is the distance on M induced by the Riemannian metric. Then $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ are injectively immersed manifolds satisfying

$$T_x\mathcal{W}^u(x) = \bigoplus_{\chi_i(x)>0} E_i(x) \quad \text{and} \quad T_x\mathcal{W}^s(x) = \bigoplus_{\chi_i(x)<0} E_i(x)$$

(see [3]). We say that μ satisfies the *almost transversality condition* if for $\mu \otimes \mu$-almost every pair $(x, y) \in M \times M$ there exist integers $p, q \in \mathbb{Z}$ and a point $z \in \mathcal{W}^u(f^p(x)) \cap \mathcal{W}^s(f^q(y))$ such that

$$T_z\mathcal{W}^u(f^p(x)) \oplus T_z\mathcal{W}^s(f^q(y)) = T_z M.$$

Recently, in [9] we gave some lower bound on the Hausdorff dimension of G_μ.

References

[1] J. F. Alves, C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.* **140** (2000), 351–398.

[2] V. Araújo and M. J. Pacifico, Large deviations for non-uniformly expanding maps. *J. Stat. Phys.* **125** (2006), 415–457.

[3] L. Barreira and Ya. B. Pesin, *Lyapunov Exponents and Smooth Ergodic Theory*, Univ. Lect. Ser. 23, Amer. Math. Soc., 2002.

[4] L. Barreira, B. Saussol and J. Schmeling, Higher-dimensional multifractal analysis, *J. Math. Pures Appl.* **81** (2002), 67–91.

[5] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.* **115** (2000), 157–193.

[6] R. Bowen, Topological entropy for noncompact sets, *Trans. Amer. Math. Soc.* **184** (1973), 125–136.

[7] D. Dolgopyat, On dynamics of mostly contracting diffeomorphisms, *Comm. Math. Phys.* **213** (2001), 181–201.

[8] D. J. Feng, K. S. Lau and J. Wu, Ergodic limits on the conformal repellers, *Adv.Math.*, **169** (2002), 58–91.

[9] M. Hirayama and N. Sumi, Hyperbolic measures with transverse intersections of stable and unstable manifolds, preprint.

[10] J. F. C. Kingman, Subadditive ergodic theory, *Ann. Probability* **1** (1973), 883–909.

[11] V. I. Oseledec, A multiplicative ergodic theorem, Lyapunov characteristic number for dynamical systems, *Trudy Moskov. Mat. Obšč.* **19** (1968), 179–210; English transl., *Trans. Mosc. Math. Soc.* **19** (1968), 197–231.

[12] C. -E. Pfister and W. G. Sullivan, On the topological entropy of saturated sets, *Ergod. Th. & Dynam. Sys.* **27** (2007), 929–956.

[13] F. Takens and E. Verbitskiy, On the variational principle for the topological entropy of certain noncompact sets, *Ergod. Th. & Dynam. Sys.* **23** (2003), 317–348.

[14] C. H. Vásquez, Statistical stability for diffeomorphisms with dominated splitting, *Ergod. Th. & Dynam. Sys.* **27** (2007), 253–283.

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